Int. J. Solids Structures Vol. 28, No. 4, pp. 491–505, 1991
Printed in Great Britain.

0020-7683/91 $3.00 + .00

PLANE PROBLEMS IN PIEZOELECTRIC MEDIA WITH DEFECTS

HORACIO SOSA
Mechanical Engineering and Mechanics Department, Drexel University, Philadelphia, PA 19104, U.S.A.

(*Received* 6 *July* 1990; *in revised form* 20 *November* 1990)

Abstract—A two-dimensional electroelastic analysis is performed on a transversely isotropic piezoelectric material containing defects. A general solution is provided in terms of complex potentials, with emphasis being placed on stress concentrations that arise in the vicinity of circular and elliptical holes. It is shown that for this genre of problem both mechanical and electrical variables are responsible for the peak stresses.

1. INTRODUCTION

For decades, piezoelectric ceramics have been the ideal materials used in the fabrication of electromechanical devices [see Pohanka and Smith (1988) for an updated review]. Their main disadvantage, however, is their brittleness: piezoceramics have a tendency to develop critical crack growth because of stress concentrations induced by both mechanical and electrical loads. Yet, defects are not limited only to cracks: voids, inclusions, delaminations and porosities may be present and contribute to failure as well. Because the new major applications of piezoelectric materials involve larger components under more severe loading conditions, there is a natural increase of the likelihood of failure. As an example one can cite the so-called "adaptive structures". It is, therefore, imperative that an analysis be developed which is capable of describing phenomena such as mechanisms that trigger crack propagation in piezoelectric media, as well as stress behavior in the vicinity of holes or inclusions.

In a recent article Sosa and Pak (1990)† study the influence that electric fields have on the distribution of stresses in the neighbourhood of a crack embedded in a transversely isotropic piezoelectric material. The analysis is carried out for the particular case of a crack with its leading edge assumed to be straight and parallel to the poling direction (or axis of transverse isotropy), as is shown in Fig. 1a. The study reveals that near the crack the stresses contained in the x–y plane are idependent of the electric field. This is not true, however, for the shear stresses in the z-direction. It is concluded in the study that electromechanical interaction is strongly influenced by the crack's orientation.

The present work has been motivated by the aforementioned article and represents an intermediate step towards developing a description of crack propagation in piezoelectric media. Towards this end we consider the same material as that referenced by Sosa and Pak. Our point of departure, however, will be two-fold: (1) the defect is no longer a crack, but a cylindrical cavity of elliptical shape; (2) the generator of the cylinder (which in the particular case of the crack becomes the crack front) is along an axis other than the axis of transverse isotropy, as represented in Fig. 1b. This new defect orientation poses mathematical difficulties not present in previous analyses which can be circumvented by resorting to a two-dimensional model. In this manner we are lead to a more complete and interesting coupling phenomenon between the mechanical and the electrical variables.

A plane strain formulation of the piezoelectric problem solved within the formalism of the complex variables technique is provided. While some work has been done in the area of fracture mechanics of piezoelectric materials, in particular from an experimental standpoint, it appears that only the work of Deeg (1980) has theoretically addressed the

† The article also provides a review of the theoretical and experimental research performed in this area.

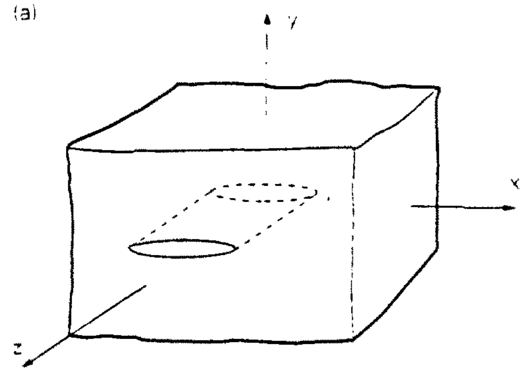

Fig. 1(a). Piezoelectric material with a crack whose leading edge is parallel to the poling direction.

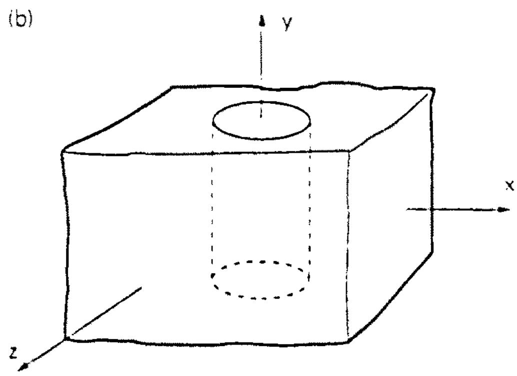

Fig. 1(b). Piezoelectric material with a cylindrical cavity whose generator is perpendicular to the poling direction.

problem of defects other than cracks.† Although the theory developed in this article is applicable to the study of the crack problem, our attention will rightly be focussed on the study of stress concentrations around elliptical and circular holes, which will include quantifying the effect that the electrical variables have on these stresses. The crack problem will be studied independently and presented elsewhere.

2. GOVERNING EQUATIONS

The theory of piezoelectricity consists of the simultaneous study of deformations and electric fields existing in anisotropic, nonconducting elastic media. The description of the piezoelectric effect is achieved by means of two mechanical and two electrical variables: the strain and stress tensors and the electric field and electric displacement vectors denoted by ε_{ij}, σ_{ij}, E_i and D_i, respectively. As a consequence, there are four possible manners of describing electromechanical interaction. In theoretical analyses it is customary to choose a representation in which the strain and electric field are the independent variables. In experimental analyses, however, constitutive relations bearing the stress and the electrical field as independent variables are preferred. In the end the choice is dictated by the particular problem that one has in mind. The present study makes use of a form in which stresses and electric displacement are the independent quantities. Thus, following Berlincourt *et al.* (1964), we write

† In contrast, the problem of a cavity embedded in an elastic isotropic dielectric has been treated more extensively. See McMeeking (1989) for references.

$$
\begin{aligned}
\varepsilon_{ij} &= s^D_{ijkl}\sigma_{kl} + g_{kij}D_k \\
E_i &= -g_{ikl}\sigma_{kl} + \beta^\sigma_{ik}D_k
\end{aligned}
\tag{1}
$$

where s^D_{ijkl} is the compliance tensor of the material measured at zero electric displacement, g_{kij} is the piezoelectric tensor, and β^σ_{ik} is the dielectric impermeability tensor measured at zero stress. Although (1) is not the most widely used form of constitutive relation, it proves to be quite convenient when formulating two-dimensional boundary value problems.

In the MKS system the aforementioned variables are measured in the following units:

$$
[\varepsilon] = \mathrm{m\,m^{-1}},\quad [\sigma] = \mathrm{N\,m^{-2}},\quad [\mathbf{E}] = \mathrm{V\,m^{-1}} = \mathrm{N\,C^{-1}},\quad [\mathbf{D}] = \mathrm{C\,m^{-2}} = \mathrm{N\,V^{-1}\,m^{-1}}
$$

$$
[\mathrm{s}^D] = \mathrm{m^2\,N^{-1}},\quad [\mathrm{g}] = \mathrm{Vm\,N^{-1}} = \mathrm{m^2\,C^{-1}},\quad [\beta^\sigma] = \mathrm{N\,m^2\,C^{-2}} = \mathrm{V^2\,N^{-1}},\quad [\phi] = \mathrm{V}
$$

where ϕ represents the electric potential given by $\mathbf{E} = -\mathrm{grad}\,\phi$. As previously mentioned in the Introduction, we will focus on transversely isotropic piezoelectrics. In such a case, and with reference to the coordinate system shown in Fig. 1, eqn (1) takes the following matrix representation:

$$
\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{zy} \\ 2\varepsilon_{zx} \\ 2\varepsilon_{xy} \end{Bmatrix}
=
\begin{bmatrix}
s_{11} & s_{12} & s_{13} & 0 & 0 & 0 \\
s_{12} & s_{11} & s_{13} & 0 & 0 & 0 \\
s_{13} & s_{13} & s_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & s_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & s_{44} & 0 \\
0 & 0 & 0 & 0 & 0 & s_{66}
\end{bmatrix}
\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{zy} \\ \sigma_{zx} \\ \sigma_{xy} \end{Bmatrix}
+
\begin{bmatrix}
0 & 0 & g_{31} \\
0 & 0 & g_{31} \\
0 & 0 & g_{33} \\
0 & g_{15} & 0 \\
g_{15} & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{Bmatrix} D_x \\ D_y \\ D_z \end{Bmatrix}
\tag{2a}
$$

where

$$
s_{66} = 2(s_{11} - s_{12})
$$

and

$$
\begin{Bmatrix} E_x \\ E_y \\ E_z \end{Bmatrix}
= -\begin{pmatrix}
0 & 0 & 0 & 0 & g_{15} & 0 \\
0 & 0 & 0 & g_{15} & 0 & 0 \\
g_{31} & g_{31} & g_{33} & 0 & 0 & 0
\end{pmatrix}
\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{zy} \\ \sigma_{zx} \\ \sigma_{xy} \end{Bmatrix}
+
\begin{pmatrix}
\beta_{11} & 0 & 0 \\
0 & \beta_{11} & 0 \\
0 & 0 & \beta_{33}
\end{pmatrix}
\begin{Bmatrix} D_x \\ D_y \\ D_z \end{Bmatrix}.
\tag{2b}
$$

From (2) it is clear that no coupling exists between the mechanical and electrical variables contained in the x–y plane. A more complete state of electromechanical interaction can be observed by reducing (1) into a two-dimensional model. Since, according to (2), the x–y plane is the isotropic plane, one can employ either the x–z or the y–z plane for the study of plane electromechanical phenomena. Choosing the former, the plane strain conditions require that†

$$
\varepsilon_{yy} = \varepsilon_{zy} = \varepsilon_{xy} = E_y = 0 \tag{3}
$$

which allows us to write

† We observe that the condition $E_y = 0$ leads to $(g_{15}^2 + \beta_{11}s_{44})D_y = 0$; however, since the quantity in parentheses is different from zero we obtain $D_y = 0$.

$$\sigma_{yy} = -\frac{1}{s_{11}}[s_{12}\sigma_{xx}+s_{13}\sigma_{zz}+g_{31}D_z]. \tag{4}$$

Substituting (3) and (4) into (2) yields the plane strain constitutive equations. To minimize notation we introduce the following definitions:

$$a_{11} = s_{11}-\frac{s_{12}^2}{s_{11}}, \quad a_{12} = s_{13}-\frac{s_{12}s_{13}}{s_{11}}, \quad a_{22} = s_{33}-\frac{s_{13}^2}{s_{11}}, \quad a_{33} = s_{44},$$

$$b_{21} = \left(1-\frac{s_{12}}{s_{11}}\right)g_{31}, \quad b_{22} = g_{33}-\frac{s_{13}}{s_{11}}g_{31}, \quad b_{13} = g_{15}, \quad \delta_{11} = \beta_{11}, \quad \delta_{22} = \beta_{33}+\frac{g_{31}^2}{s_{11}} \tag{5}$$

which are known as the *reduced material constants*. An additional step towards compactness in notation is achieved by renaming the coordinates such that $x \to x_1$ and $z \to x_2$. Hence, the two-dimensional constitutive equations can now be written as

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{12} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} + \begin{pmatrix} 0 & b_{21} \\ 0 & b_{22} \\ b_{13} & 0 \end{pmatrix} \begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix} \tag{6a}$$

$$\begin{Bmatrix} E_1 \\ E_2 \end{Bmatrix} = -\begin{pmatrix} 0 & 0 & b_{13} \\ b_{21} & b_{22} & 0 \end{pmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} + \begin{pmatrix} \delta_{11} & 0 \\ 0 & \delta_{22} \end{pmatrix} \begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix}. \tag{6b}$$

For a complete formulation of the piezoelectric problem we need to supplement (6) with the equations of elastic equilibrium and Gauss' Law of Electrostatics, which in two dimensions and in the absence of body forces and free electric volume charge are given by

$$\frac{\partial \sigma_{11}}{\partial x_1}+\frac{\partial \sigma_{12}}{\partial x_2} = 0, \quad \frac{\partial \sigma_{12}}{\partial x_1}+\frac{\partial \sigma_{22}}{\partial x_2} = 0, \quad \frac{\partial D_1}{\partial x_1}+\frac{\partial D_2}{\partial x_2} = 0. \tag{7a–c}$$

Furthermore, the strain and electric field components satisfy the compatibility relations

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2}+\frac{\partial^2 \varepsilon_{22}}{\partial x_1^2}-2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} = 0, \quad \frac{\partial E_1}{\partial x_2}-\frac{\partial E_2}{\partial x_1} = 0. \tag{8a,b}$$

The solution to the system of equations furnished by (6)–(8) is sought by means of a stress function $U(x_1, x_2)$ which satisfies the elastic equilibrium equations when defined as

$$\sigma_{11} = \frac{\partial^2 U}{\partial x_2^2}, \quad \sigma_{22} = \frac{\partial^2 U}{\partial x_1^2}, \quad \sigma_{12} = -\frac{\partial^2 U}{\partial x_1 \partial x_2}. \tag{9}$$

In addition, we introduce an induction function $\psi(x_1, x_2)$ such that

$$D_1 = \frac{\partial \psi}{\partial x_2}, \quad D_2 = -\frac{\partial \psi}{\partial x_1} \tag{10}$$

which satisfies (7c). Next, introducing (9) and (10) into (6a), and later into (8a) leads to

$$a_{11}U_{,2222}+a_{22}U_{,1111}+(2a_{12}+a_{33})U_{,1122}-(b_{21}+b_{13})\psi_{,122}-b_{22}\psi_{,111} = 0 \tag{11}$$

where the commas denote differentiation. Similarly, substituting (9) and (10) successively into (6b) and (8b) yields

$$(b_{13}+b_{21})U_{,122}+b_{22}U_{,111}+\delta_{22}\psi_{,11}+\delta_{11}\psi_{,22} = 0. \tag{12}$$

Equations (11) and (12) can be expressed in compact form by writing:

$$\begin{aligned} L_4U - L_3\psi &= 0 \\ L_3U + L_2\psi &= 0 \end{aligned} \tag{13}$$

where L_i ($i = 4, 3, 2$) are differential operators of order four, three, and two, reflecting the elastic, piezoelectric, and dielectric properties of the material, respectively, and given by

$$L_4 = a_{22}\frac{\partial^4}{\partial x_1^4} + a_{11}\frac{\partial^4}{\partial x_2^4} + (2a_{12}+a_{33})\frac{\partial^4}{\partial x_1^2\partial x_2^2}$$

$$L_3 = b_{22}\frac{\partial^3}{\partial x_1^3} + (b_{21}+b_{13})\frac{\partial^3}{\partial x_1\partial x_2^2}, \quad L_2 = \delta_{22}\frac{\partial^2}{\partial x_1^2} + \delta_{11}\frac{\partial^2}{\partial x_2^2}. \tag{14}$$

Thus, the plane piezoelectric problem is governed by a system of two partial differential equations coupled in U and ψ. If we eliminate ψ, (13) is reduced to a single sixth order partial differential equation for the stress function, namely

$$(L_4L_2 + L_3L_3)U = 0 \tag{15}$$

or written explicitly

$$\left\{(a_{22}\delta_{22}+b_{22}^2)\frac{\partial^6}{\partial x_1^6} + a_{11}\delta_{11}\frac{\partial^6}{\partial x_2^6} + (a_{22}\delta_{11}+2a_{12}\delta_{22}+a_{33}\delta_{22}+2b_{21}b_{22}+2b_{13}b_{22})\frac{\partial^6}{\partial x_1^4\partial x_2^2}\right.$$
$$\left. + (a_{11}\delta_{22}+2a_{12}\delta_{11}+a_{33}\delta_{11}+b_{21}^2+b_{13}^2+2b_{21}b_{13})\frac{\partial^6}{\partial x_1^2\partial x_2^4}\right\}U = 0. \tag{16}$$

Equation (16) can be solved by means of complex variables†: we express the solution by means of a function $U(z)$ defined as

$$U(z) = U(x_1+\mu x_2), \quad \mu = \alpha + i\beta, \quad i = \sqrt{-1} \tag{17}$$

where z is a generalized complex variable, μ is a complex parameter, and α and β are real numbers. By introducing (17) into (16), and using the chain rule of differentiation, an expression of the form $\{\cdot\}U^{VI} = 0$ is obtained. A nontrivial solution follows by setting the characteristic equation (that is, the quantity enclosed within braces) equal to zero, namely

$$a_{11}\delta_{11}\mu^6 + (a_{11}\delta_{22}+2a_{12}\delta_{11}+a_{33}\delta_{11}+b_{21}^2+b_{13}^2+2b_{21}b_{13})(\mu^4+(a_{22}\delta_{11}+2a_{12}\delta_{22}+a_{33}\delta_{22}$$
$$+2b_{21}b_{22}+2b_{13}b_{22})\mu^2+(a_{22}\delta_{22}+b_{22}^2) = 0. \tag{18}$$

Owing to the particular material symmetry of the piezoelectric under investigation, the polynomial is expressed in terms of even powers of μ. This allows us to solve (18) analytically, rendering

$$\mu_1 = \beta_1 i, \quad \mu_2 = \alpha_2+i\beta_2, \quad \mu_3 = -\alpha_2+i\beta_2, \quad \mu_4 = \bar{\mu}_1, \quad \mu_5 = \bar{\mu}_2, \quad \mu_6 = \bar{\mu}_3 \tag{19}$$

where β_1, α_2 and β_2 depend on the material constants. Once the roots μ_k, $k = 1, 2, 3$ are known, the solution is written as

† We extend the ideas developed by Lekhnitskii (1981) in the framework of anisotropic elasticity.

$$U(x_1, x_2) = 2\mathscr{R} \sum_{k=1}^{3} U_k(z_k) \tag{20}$$

where

$$z_k = x_1 + \mu_k x_2 = (x_1 + \alpha_k x_2) + i\beta_k x_2 \tag{21}$$

and $\mathscr{R}$ denotes the real part of a given complex expression. The next step is to find the function ψ using one of the equations (13). If we consider $L_3 U = -L_2 \psi$, assuming solutions of the form $U(x_1 + \mu_k x_2)$ and $\psi(x_1 + \mu_k x_2)$, we obtain

$$b(\mu_k) U_k''' = -\delta(\mu_k) \psi_k'' \tag{22}$$

where

$$b(\mu_k) = (b_{21} + b_{13})\mu_k^2 + b_{22}, \quad \delta(\mu_k) = \delta_{11}\mu_k^2 + \delta_{22}. \tag{23}$$

Integration of (22) yields†

$$\psi_k(z_k) = \lambda_k U_k'(z_k) \tag{24}$$

where

$$\lambda_k(\mu_k) = -\frac{b(\mu_k)}{\delta(\mu_k)}, \quad \delta(\mu_k) \neq 0. \tag{25}$$

Thus, the solution for the electric induction becomes

$$\psi(x_1, x_2) = 2\mathscr{R} \sum_{k=1}^{3} \psi_k(z_k) = 2\mathscr{R} \sum_{k=1}^{3} \lambda_k U_k'(z_k). \tag{26}$$

Alternatively, we could have obtained ψ by using $L_4 U = L_3 \psi$, leading to

$$\psi_k = \bar{\lambda}_k U_k', \quad \bar{\lambda}_k(\mu_k) = \frac{a(\mu_k)}{b(\mu_k)}, \quad b(\mu_k) \neq 0, \quad a(\mu_k) = a_{11}\mu_k^4 + (2a_{12} + a_{33})\mu_k^2 + a_{22}.$$

But $\lambda_k(\mu_k) = \bar{\lambda}_k(\mu_k)$, since by (18), $a(\mu_k)\delta(\mu_k) + b^2(\mu_k) = 0$; hence the same function ψ_k is obtained.

With the aid of (20) and (26) we can write expressions for the stress and electric displacement components. Towards this end and in order to reduce the order of the derivatives, it is convenient to introduce new functions φ_k of the complex variable z (hereafter called the complex potentials) which are defined as

$$\varphi_k(z_k) = U_k' = \frac{dU_k}{dz_k} \tag{27}$$

where $k = 1, 2, 3$, and no summation is implied over repeated indices. The use of (9), (20) and (27) leads to the stress components

† The arbitrary constants of integration can be set equal to zero. If they are retained, they can be embedded in the linear and constant terms of eqn (44).

$$\sigma_{11} = 2\mathscr{R}\sum_{k=1}^{3}\mu_k^2\varphi_k'(z_k),\quad \sigma_{22} = 2\mathscr{R}\sum_{k=1}^{3}\varphi_k'(z_k),\quad \sigma_{12} = -2\mathscr{R}\sum_{k=1}^{3}\mu_k\varphi_k'(z_k). \tag{28}$$

Likewise (10), (26) and (27) yield

$$D_1 = 2\mathscr{R}\sum_{k=1}^{3}\lambda_k\mu_k\varphi_k'(z_k),\quad D_2 = -2\mathscr{R}\sum_{k=1}^{3}\lambda_k\varphi_k'(z_k). \tag{29}$$

Finally, using the constitutive equations in conjunction with (28) and (29) allows us to find expressions for the elastic displacement, the electric field and the electric potential. The results are summarized below.

The components of strain result in

$$\begin{aligned}\varepsilon_{11} &= 2\mathscr{R}\sum_{k=1}^{3}\{a_{11}\mu_k^2+a_{12}-b_{21}\lambda_k\}\varphi_k' \\ \varepsilon_{22} &= 2\mathscr{R}\sum_{k=1}^{3}\{a_{12}\mu_k^2+a_{22}-b_{22}\lambda_k\}\varphi_k' \\ 2\varepsilon_{12} &= 2\mathscr{R}\sum_{k=1}^{3}\{-a_{33}\mu_k+b_{13}\lambda_k\mu_k\}\varphi_k'.\end{aligned} \tag{30}$$

Using the strain displacement relationship

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j}+u_{j,i}) \tag{31}$$

the integration of the normal strains renders

$$u_1 = 2\mathscr{R}\sum_{k=1}^{3}p_k\varphi_k(z_k)+\omega x_2+u_0,\quad u_2 = 2\mathscr{R}\sum_{k=1}^{3}q_k\varphi_k(z_k)-\omega x_1+v_0 \tag{32}$$

where the constants ω, u_0 and v_0 represent rigid body displacements and

$$p_k = a_{11}\mu_k^2+a_{12}-b_{21}\lambda_k,\quad q_k = \frac{a_{12}\mu_k^2+a_{22}-b_{22}\lambda_k}{\mu_k}. \tag{33}$$

Similarly, using (28) and (29) in conjunction with (6b) gives the components of the electric field:

$$E_1 = 2\mathscr{R}\sum_{k=1}^{3}\{b_{13}+\delta_{11}\lambda_k\}\mu_k\varphi_k',\quad E_2 = -2\mathscr{R}\sum_{k=1}^{3}\{b_{21}\mu_k^2+b_{22}+\delta_{22}\lambda_k\}\varphi_k'. \tag{34}$$

Finally, integration of $\mathbf{E} = -\text{grad}\,\phi$ leads to the electric potential:

$$\phi = -2\mathscr{R}\sum_{k=1}^{3}\{b_{13}+\delta_{11}\lambda_k\}\mu_k\varphi_k+\phi_0 \tag{35}$$

where ϕ_0 is a reference potential.

Recapitulating, the plane strain piezoelectric problem has been reduced to one of finding three complex potentials, φ_1, φ_2 and φ_3, in some region Ω of the medium. Each potential is a function of a different generalized complex variable $z_k = x_1+\mu_k x_2$. Alternatively, the complex potentials can be viewed as functions of the ordinary complex variable $z_k = x_1^{(k)}+ix_2^{(k)}$ where

$$x_1^{(k)} = x_1 + \alpha_k x_2, \quad x_2^{(k)} = \beta_k x_2. \tag{36}$$

Using this point of view the functions φ_1, φ_2 and φ_3 must be determined in regions Ω_1, Ω_2 and Ω_3, respectively, obtained from Ω by the affine transformations (36).

We should note, however, that the problem as formulated is still undetermined. The complex potentials need to be determined subject to certain boundary and jump conditions on the boundary (or surfaces of discontinuity) $\partial\Omega$. The piezoelectric boundary conditions are of mechanical nature (prescribed elastic displacement $\bar{\mathbf{u}}$ or surface traction $\bar{\mathbf{t}}$) and of electrical nature (prescribed electric field or electric displacement). Thus, calling $\partial\Omega_t$, $\partial\Omega_u$, $\partial\Omega_D$ and $\partial\Omega_\phi$ the parts of the boundary $\partial\Omega$ where $\mathbf{t}$, $\mathbf{u}$, $\mathbf{D}$ and ϕ are prescribed, we can write in the most general case (see Eringen and Maugin, 1989)

$$\begin{aligned} \boldsymbol{\sigma}\mathbf{n} &= \bar{\mathbf{t}} \quad \text{on } \partial\Omega_t \\ \mathbf{u} &= \bar{\mathbf{u}} \quad \text{on } \partial\Omega_u \\ \mathbf{n}\cdot[\![\mathbf{D}]\!] &= w_e \quad \text{on } \partial\Omega_D \\ [\![\phi]\!] &= 0 \quad \text{on } \partial\Omega_\phi \end{aligned} \tag{37a–d}$$

where w_e is a prescribed surface charge density and $\mathbf{n}$ is the outward unit normal to $\partial\Omega$. We note that (37d) is a consequence of

$$\mathbf{n}\times[\![\mathbf{E}]\!] = \mathbf{0} \quad \text{and} \quad \mathbf{E} = -\text{grad}\,\phi.$$

If we impose boundary and jump conditions in terms of $\mathbf{t}$ and $\mathbf{D}$ only (as will be done in the present study), we can write

$$\frac{\partial U}{\partial x_1} = -\int_0^s t_2\,\mathrm{d}s, \quad \frac{\partial U}{\partial x_1} = -\int_0^s t_1\,\mathrm{d}s, \quad \psi = -\int_0^s D_n\,\mathrm{d}s \tag{38}$$

where t_1 and t_2 are the rectangular Cartesian components of $\mathbf{t}$, D_n is the normal component of $\mathbf{D}$, and $\mathrm{d}s$ is an element of arc length on $\partial\Omega$. Or in terms of the complex potentials we can rewrite (38) as

$$\begin{aligned} 2\mathcal{R}\sum_{k=1}^{3}\varphi_k(z_k) &= -\int_0^s t_2\,\mathrm{d}s \\ 2\mathcal{R}\sum_{k=1}^{3}\mu_k\varphi_k(z_k) &= \int_0^s t_1\,\mathrm{d}s \\ 2\mathcal{R}\sum_{k=1}^{3}\lambda_k\varphi_k(z_k) &= -\int_0^s D_n\,\mathrm{d}s. \end{aligned} \tag{39}$$

3. INFINITE PIEZOELECTRIC MEDIUM WITH AN ELLIPTICAL CAVITY

Consider an infinite space filled with transversely isotropic piezoelectric material and containing a hole of elliptical shape. The axes of the cavity of length $2a$ and $2b$ are assumed

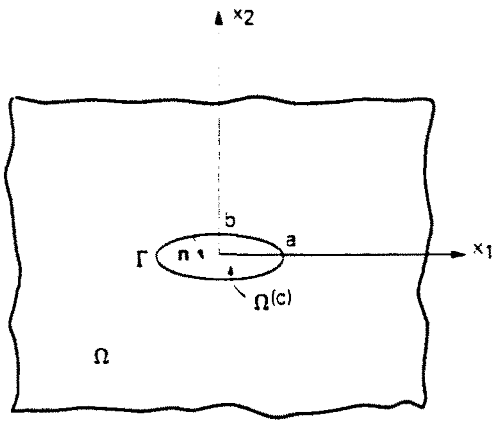

Fig. 2. Elliptical hole in an infinite piezoelectric medium.

to be placed along the axes of elastic symmetry of the material as shown in Fig. 2. Furthermore, call Γ the boundary of the hole with outward unit normal **n**. Mechanical or electrical loads applied at remote distances from the hole induce deformations and electric fields in the region of space Ω filled with matter described by (6). The induced or applied fields also exist in the region inside the cavity $\Omega^{(c)}$ (filled with vacuum) described by the constitutive relation

$$\mathbf{D}^{(c)} = \varepsilon_0 \mathbf{E}^{(c)} \tag{40}$$

where ε_0 is the dielectric constant (or permittivity) of vacuum ($\varepsilon_0 = 8.85 \times 10^{-12}$ N V^{-2}). In $\Omega^{(c)}$ the governing equation is simply the two-dimensional Laplace equation:

$$\nabla^2 \phi^{(c)} = 0 \tag{41}$$

and the normal component of the electric displacement can be expressed as

$$\mathbf{D}^{(c)} \cdot \mathbf{n} = -\varepsilon_0 \frac{\partial \phi^{(c)}}{\partial \mathrm{n}}.$$

The depicted situation constitutes a two-domain boundary value problem. Hence, once the electric displacement and electric potential are found in both Ω and $\Omega^{(c)}$, the electric boundary conditions become

$$\begin{aligned} D_n &= -\varepsilon_0 \frac{\partial \phi^{(c)}}{\partial n} + w_e \\ \phi &= \phi^{(c)} \end{aligned} \quad \text{on } \Gamma \tag{42}$$

where the quantities on the left-hand side of (42) are evaluated within the piezoelectric. A significant simplification to the original two-domain problem is achieved by neglecting the surroundings (the vacuum in this case) of Ω. This process is permissible because the dielectric constants in the piezoelectric material are significantly larger than ε_0. Consequently, assuming that $w_e = 0$, the boundary conditions at the surface of a traction-free cavity can be expressed as (see also Pak, 1990)

$$\begin{aligned} \mathbf{t} &= \mathbf{0} \\ \mathbf{D} \cdot \mathbf{n} &= 0 \end{aligned} \quad \text{on } \Gamma. \tag{43}$$

The problem is now merely reduced to one of finding the complex potentials in the region Ω. Towards this end we assume a general solution of the form

$$\varphi_k(z_k) = (A_k + iA_k^*)\log z_k + (B_k + iB_k^*)z_k + \varphi_k^0(z_k), \quad k = 1, 2, 3 \tag{44}$$

where A_k, A_k^*, B_k, B_k^* are real constants and

$$\varphi_k^0(z_k) = \sum_{n=0}^{\infty} \frac{a_n^{(k)}}{z_k^n} = a_0^{(k)} + \frac{a_1^{(k)}}{z_k} + \cdots, \quad k = 1, 2, 3 \tag{45}$$

is a holomorphic function up to infinity with real coefficients $a_n^{(k)}$. The boundary conditions enforced at the rim of the hole require that $A_k + iA_k^* = 0$ for φ_k to be single valued. Furthermore, the constant B_k and B_k^* are determined from the far field loading conditions, as is described at the end of this section.

To find the holomorphic functions we make use of (39), with their right-hand sides set equal to zero. That is

$$2\mathscr{R}\sum_{k=1}^{3}\varphi_k = 0, \quad 2\mathscr{R}\sum_{k=1}^{3}\mu_k\varphi_k = 0 \quad 2\mathscr{R}\sum_{k=1}^{3}\lambda_k\varphi_k = 0 \tag{46}$$

or substituting the general expression for φ_k:

$$\begin{aligned}
2\mathscr{R}\sum_{k=1}^{3}\varphi_k^0 &= -2\mathscr{R}\sum_{k=1}^{3}(B_k + iB_k^*)z_k \\
2\mathscr{R}\sum_{k=1}^{3}\mu_k\varphi_k^0 &= -2\mathscr{R}\sum_{k=1}^{3}(B_k + iB_k^*)\mu_k z_k \\
2\mathscr{R}\sum_{k=1}^{3}\lambda_k\varphi_k^0 &= -2\mathscr{R}\sum_{k=1}^{3}(B_k + iB_k^*)\lambda_k z_k.
\end{aligned} \tag{47}$$

Equation (47) can be solved for φ_k^0 by means of a conformal transformation which maps the exterior of three ellipses (one for each root μ_k) contained in the z_k plane into the exterior of the unit circle (of boundary γ) located in the ζ_k-plane. The relevant transformation is (see Lekhnitskii, 1981)

$$z_k = \frac{a - i\mu_k b}{2}\zeta_k + \frac{a + i\mu_k b}{2}\frac{1}{\zeta_k}. \tag{48}$$

Note that both z_k and ζ_k travel on Γ and γ, respectively, in a counterclockwise sense. Furthermore, the three points on the contours of Ω_k map into a single point on the contour of the unit circle, which is described by $\zeta_k = \sigma = e^{i\theta}$, $0 \leqslant \theta \leqslant 2\pi$. Assigning a notation of $\Phi_k^0(\zeta_k)$ to the functions $\varphi_k^0(z_k)$ after (48) is applied, the boundary conditions (47) become

$$\begin{aligned}
\sum_{k=1}^{3}\Phi_k^0(\sigma) + \bar{\Phi}_k^0(\bar{\sigma}) &= \bar{l}_1\sigma + l_1\sigma^{-1} \\
\sum_{k=1}^{3}\mu_k\Phi_{(k)}^0(\sigma) + \bar{\mu}_k\bar{\Phi}_k^0(\bar{\sigma}) &= \bar{l}_2\sigma + l_2\sigma^{-1} \\
\sum_{k=1}^{3}\lambda_k\Phi_k^0(\sigma) + \bar{\lambda}_k\bar{\Phi}_k^0(\bar{\sigma}) &= \bar{l}_3\sigma + l_3\sigma^{-1}
\end{aligned} \tag{49}$$

where

$$\begin{aligned}
l_1 &= \sum_{k=1}^{3}[-aB_k + i(\beta_k B_k^* - \alpha_k B_k)b] \\
l_2 &= \sum_{k=1}^{3}\{(\beta_k B_k^* - \alpha_k B_k)a + i[(\beta_k^2 - \alpha_k^2)B_k + 2\alpha_k\beta_k B_k^*]b\} \\
l_3 &= \sum_{k=1}^{3}\{(I_k B_k^* - R_k B_k)a + i[(I_k\beta_k - R_k\alpha_k)B_k + (I_k\alpha_k + R_k\beta_k)B_k^*]b\}
\end{aligned} \tag{50}$$

with $\bar{l}_1$, $\bar{l}_2$ and $\bar{l}_3$ being their complex conjugates. To solve (49) for the functions Φ_k^0 we multiply both sides by $d\sigma\,\sigma-\zeta$ and integrate over γ, where ζ is any point outside the unit circle. By observing that

$$\int_\gamma \frac{\Phi_k^0(\sigma)}{\sigma-\zeta_k}\,d\sigma = -2\pi i\Phi_k^0(\zeta_k),\quad \int_\gamma \frac{\bar{\Phi}_k^0(\bar{\sigma})}{\sigma-\zeta_k}\,d\sigma = 0,\quad \int_\gamma \frac{\sigma\,d\sigma}{\sigma-\zeta} = 0,\quad \int_\gamma \frac{d\sigma}{(\sigma-\zeta)\sigma} = -2\pi i\frac{1}{\zeta} \tag{51}$$

we obtain

$$\begin{pmatrix} 1 & 1 & 1 \\ \mu_1 & \mu_2 & \mu_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix} \begin{Bmatrix} \Phi_1^0 \\ \Phi_2^0 \\ \Phi_3^0 \end{Bmatrix} = \begin{Bmatrix} l_1 \\ l_2 \\ l_3 \end{Bmatrix} \frac{1}{\zeta}. \tag{52}$$

Solving for the function Φ_k^0, and regarding ζ as ζ_k when k takes the values 1, 2 or 3, we can express the solution as

$$\Phi_k^0 = [\Lambda_{k1}l_1+\Lambda_{k2}l_2+\Lambda_{k3}l_3]\frac{1}{\zeta_k},\quad k = 1,2,3 \tag{53}$$

where Λ_{11}, Λ_{12}, etc., are the elements of the matrix

$$\Lambda = [\Lambda_{kl}] = \frac{1}{\Delta}\begin{pmatrix} \mu_2\lambda_3-\mu_3\lambda_2 & \lambda_2-\lambda_3 & \mu_3-\mu_2 \\ \mu_3\lambda_1-\mu_1\lambda_3 & \lambda_3-\lambda_1 & \mu_1-\mu_3 \\ \mu_1\lambda_2-\mu_2\lambda_1 & \lambda_1-\lambda_2 & \mu_2-\mu_1 \end{pmatrix} \tag{54}$$

$$\Delta = (\lambda_2-\lambda_3)\mu_1+(\lambda_3-\lambda_1)\mu_2+(\lambda_1-\lambda_2)\mu_3. \tag{55}$$

Finally, to obtain $\varphi_k^0(z_k)$, (53) is inverted by substituting each ζ_k by

$$\zeta_k = \frac{z_k+\sqrt{z_k^2-(a^2+\mu_k^2b^2)}}{a-i\mu_kb} = \frac{a+i\mu_kb}{z_k-\sqrt{z_k^2-(a^2+\mu_k^2b^2)}} \tag{56}$$

yielding the three complex potentials

$$\varphi_k(z_k) = (B_k+iB_k^*)z_k+[\Lambda_{k1}l_1+\Lambda_{k2}l_2+\Lambda_{k3}l_3]\frac{z_k-\sqrt{z_k^2-(a^2+\mu_k^2b^2)}}{a+i\mu_kb}. \tag{57}$$

To determine $\boldsymbol{\sigma}$ and $\mathbf{D}$, the derivative of (57) with respect to z_k is evaluated, which results in

$$\varphi_k'(z_k) = (B_k+iB_k^*)+[\Lambda_{k1}l_1+\Lambda_{k2}l_2+\Lambda_{k3}l_3]\frac{1}{a+i\mu_kb}\left\{1-\frac{z_k}{\sqrt{z_k^2-(a^2+\mu_k^2b^2)}}\right\}. \tag{58}$$

To solve for the constants B_k, B_k^*, one must invoke the remote electromechanical loading conditions. In the most general case, three mechanical and two electrical variables can be enforced. If these loads are the stress and electric displacement components, by making use of (28), (29) and (58) when $|z| \to \infty$, a system of five equations in the six unknowns B_k, B_k^* is obtained. Without loss of generality one can arbitrarily set one of these constants equal to zero. Thus, in the remainder of this article it is assumed that $B_1^* = 0$. Otherwise, a sixth equation can be formulated in terms of the remote rigid rotation. That is, one can impose the condition $2\omega = u_{2,1}-u_{1,2} = 0$, as $|z| \to \infty$. Once explicit solutions for B_k and B_k^* are obtained in terms of the remote load and material properties, it is a simple matter to show that (50) renders

$$l_1 = -\frac{a\sigma_{22}^{(\infty)}}{2} + i\frac{b\sigma_{12}^{(\infty)}}{2}, \quad l_2 = \frac{a\sigma_{12}^{(\infty)}}{2} - i\frac{b\sigma_{11}^{(\infty)}}{2}, \quad l_3 = \frac{aD_2^{(\infty)}}{2} - i\frac{bD_1^{(\infty)}}{2}. \tag{59}$$

That is, l_1, l_2 and l_3 depend on the applied load and the geometry of the cavity. In the following section, the stress distribution around elliptical and circular holes will be found in terms of the complex potentials given by (57).

4. EXAMPLES

In this section it is assumed that the piezoelectric medium is a PZT-4 ceramic with material constants that can be found in Berlincourt *et al.* (1964). The corresponding reduced material constants obtained from (5) are

$$a_{11} = 8.205 \times 10^{-12}, \quad a_{12} = -3.144 \times 10^{-12}, \quad a_{22} = 7.495 \times 10^{-12},$$

$$a_{33} = 19.3 \times 10^{-12}\,(\mathrm{m^2\,N^{-1}})$$

$$b_{21} = -16.62 \times 10^{-3}, \quad b_{22} = 23.96 \times 10^{-3}, \quad b_{13} = 39.4 \times 10^{-3}\,(\mathrm{m^2\,C^{-1}})$$

$$\delta_{11} = 7.66 \times 10^{7}, \quad \delta_{22} = 9.82 \times 10^{7}\,(\mathrm{V^2\,N^{-1}}).$$

To appreciate the order of magnitude of the variables involved in this type of problem we note that the poling process in ceramics (that is, the process through which the piezoelectric effect is induced) takes place at electric field levels of 10^6 V m^{-1}. Furthermore, typical applications involve electrical displacements of the order of 10^{-3} to 10^{-2} C m^{-2}, while the stresses can vary between 10^6 and 10^7 N m^{-2}.

As a closure for the theory developed in the previous sections two examples are presented which can be regarded as of fundamental importance in drawing conclusions about the theory of electroelasticity with defects.

Example 1. *The elastic and electroelastic cases*

Quite often it has been claimed that stress analyses in piezoelectrics can be implemented neglecting the effect of the electrical variables. As evidence, note that it is not unusual to find fracture mechanics analyses in piezoelectricity based on conventional approaches drawn from the theory of elasticity. While this approach certainly simplifies the study, it may also produce misleading results. Moreover, discrepancies tend to become more pronounced for stress levels near crack tips or cavities. The purpose of this example is to exhibit the differences that may arise when using a purely elastic model versus an electroelastic model.

Consider the elliptical cavity shown in Fig. 2 with boundary conditions given by (43). For simplification we consider far field mechanical loading in the x_1-direction: $\sigma_{11}^{(\infty)} = p$. We look for the stresses along the x_2-axis, and, more specifically, the maximum values at $x_2 = b$.

If in (6) we neglect the terms containing the electrical variables, the problem is reduced to one of purely anisotropic elasticity governed by $L_4U = 0$. Using a procedure similar to the one described in Section 2, we look for two complex potentials and subsequently compute the stress component σ_{11}. The results are displayed in the second column of Table 1 for four different ratios of a/b.

Table 1. Values of $(\sigma_{11})_{max}/p$ (at $x_2 = b$)

a/b	Elastic	Electroelastic	% Difference
3	1.622	1.743	7.5
1	2.870	3.230	12.5
1/3	6.610	7.700	16.5
1/10	19.7	23.26	18.0

If under the same loading conditions the electrical terms are retained, the problem falls in the domain of the theory described in Sections 2 and 3. Solving for the three potentials and the corresponding stresses for this electroelastic case leads to the results shown in the third column of Table 1. The last column in the table provides the percentage differences between the purely elastic and electroelastic cases at the point of maximum stress. Note that these differences are by no means negligible, which clearly indicates that a stress analysis should take account of both electrical and mechanical effects.

Example 2. The circular hole

We use this simple configuration to illustrate stress and electric field variations at the rim of the hole when remote mechanical or electrical load is applied in the x_2-direction. In this case it is convenient to introduce polar coordinates. Thus, letting $z_k = r(\cos\theta + \mu_k \sin\theta)$, $r \geqslant a$, $0 \leqslant \theta < 2\pi$, the applied far field load, the boundary conditions and the complex potentials can be expressed as

$$\sigma_{22}^{(\infty)} = \sigma_0, \quad \text{or} \quad D_2^{(\infty)} = D_0$$

$$\sigma_r = \sigma_\theta = D_r = 0 \quad \text{on} \quad r = a$$

$$\varphi_k(r,\theta) = (B_k + iB_k^*) r(\cos\theta + \mu_k \sin\theta) + [\Lambda_{k1} l_1 + \Lambda_{k2} l_2 + \Lambda_{k3} l_3] \frac{r}{a} (\cos\theta - i\sin\theta).$$

Figures 3 and 4 depict the distribution of σ_θ and D_θ, respectively, normalized with respect to the applied load. The following observations are made: (1) because of anisotropy

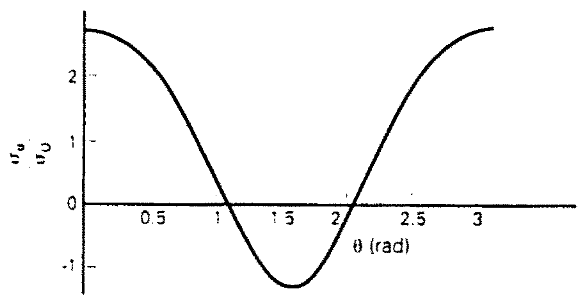

Fig. 3. σ_θ variation on the rim of a circular hole subjected to remote mechanical loading.

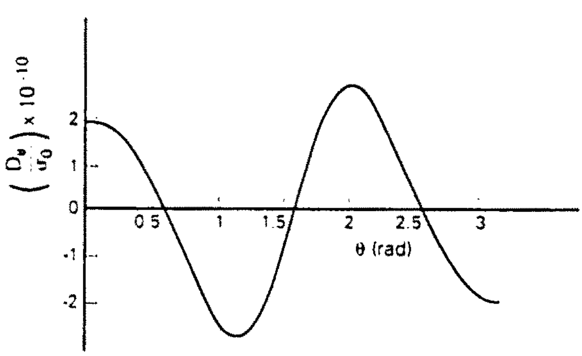

Fig. 4. D_θ variation on the rim of a circular hole subjected to remote mechanical loading.

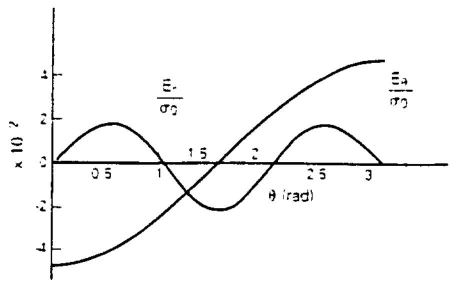

Fig. 5. E_r and E_θ variations on the rim of a circular hole subjected to remote mechanical loading.

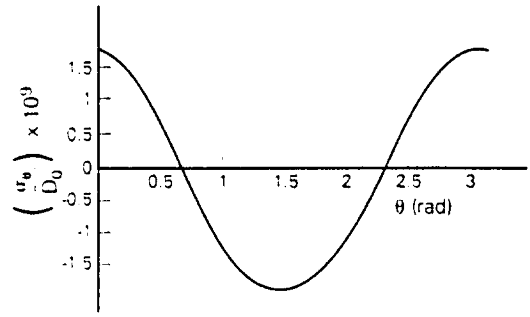

Fig. 6. σ_θ variation on the rim of a circular hole subjected to remote electrical loading.

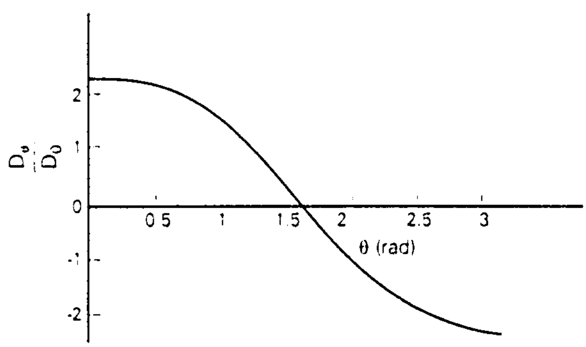

Fig. 7. D_θ variation on the rim of a circular hole subjected to remote electrical loading.

the maximum value of the hoop stress is almost 16% less than the maximum stress when a load in the x_1-direction is applied (see Table 1); (2) the maximum values of σ_θ occur at $\theta = 0°, 180°$, while D_θ reaches its maximums at $\theta = 65°, 114°$. The components of the induced electric field are shown in Fig. 5. Observe that according to the present normalization, applied stresses of order 10^7 N m^{-1} induce electric displacements of order 10^{-3} C m^{-2} and electric fields of order 10^5 V m^{-1}.

When an electrical load in the form of D_0 is applied it will produce stresses and an electric field. Figures 6 and 7 show the normalized values of σ_θ and D_θ, respectively, while Fig. 8 represents the distribution of the components of the electric field. It is clear that an applied electric displacement of order 10^{-3} C m^{-2} produces stresses of order 10^6 N m^{-2} and electric fields of order 10^5 V m^{-1}. Furthermore, we note that while D_θ and E_θ are maximum at $\theta = 0°, 180°$, σ_θ achieves its maximum at $\theta = 90°$.

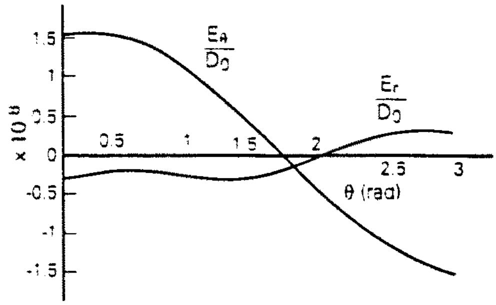

Fig. 8. E_r and E_θ variations on the rim of a circular hole subjected to remote electrical loading.

Obviously, other loading conditions can be analysed. At this point, however, it is of more fundamental importance to pursue qualitative results, rather than present a collection of diverse loading and geometric configurations.

5. CONCLUSIONS

A plane strain piezoelectric problem has been formulated and solved by means of complex variables theory. The analysis shows that stresses, displacements, electric field components, etc., can be expressed in terms of three complex potentials.

A derivation of these potentials has been illustrated by means of a problem in which an elliptical cavity is embedded in an infinite piezoelectric medium. Within this context, possible loading and boundary conditions have also been discussed. Furthermore, it has been shown that stress analyses in the vicinity of a hole can produce incorrect results if electrical effects are not taken into account.

Acknowledgement—The author wishes to acknowledge that this work was supported by the National Science Foundation under Grant No. MSS-8910374.

REFERENCES

Berlincourt, D. A., Curran, D. R. and Jaffe, H. (1964). Piezoelectric and piezoceramic materials and their function in transducers. In *Physical Acoustics* (Edited by W. P. Mason), Vol. I A. Academic Press, New York.

Deeg, W. F. (1980). The analysis of dislocation, crack and inclusion problems in piezoelectric solids. Ph.D. Thesis, Stanford University, CA.

Eringen, A. C. and Maugin, G. A. (1989). *Electrodynamics of Continua* I. Springer, Berlin.

Lekhnitskii, S. G. (1981). *Theory of Elasticity of an Anisotropic Body*, English translation. Mir Publishers, Moscow.

McMeeking, R. M. (1989). Electrostrictive stresses near crack-like flaws. *Z. Angew. Math. Phys.* **40**, 615–627.

Pak, Y. E. (1990). Crack extension force in a piezoelectric material. *ASME J. Appl. Mech.* **57**, 647–653.

Pohanka, R. C. and Smith, P. L. (1988). *Electronic Ceramics, Properties, Devices and Applications* (Edited by L. N. Levinson), Chapter 2. Marcel Dekker, New York.

Sosa, H. A. and Pak, Y. E. (1990). Three-dimensional eigenfunction analysis of a crack in a piezoelectric material. *Int. J. Solids Structures* **26**, 1–15.